What is a Napierian Logarithm?

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Raymond Ayoub

§1. INTRODUCTION. The invention of logarithms in 1614 by John Napier, baron of Merchiston in Scotland, is one of those rare parthogenic events in the history of science—there seemed to be no visible developments which foreshadowed its creation. The subsequent progress completely revolutionized arithmetic calculations in various areas of science, especially in astronomy. It is startling to realize that the spectacular, if not miraculous, development of computers in the last two decades has rendered tables of logarithms, and the portable version—the slide rule—essentially obsolete.

This was not always so. A generation ago, the use of tables of logarithms was an integral part of secondary education. A student had to learn the meaning of the terms logarithm, base, antilogarithm, mantissa, and interpolation and had to learn to use the tables. Moreover, the tables were either to base 10, earlier called "Briggsian" or to base e earlier called "hyperbolic" and before that "Napierian". The logarithms invented by Napier were closely allied to, but not the same as, hyperbolic.

Over the years, various authors have vied with one another to produce tables of greater precision as well as ease of use. Indeed as recently as 1964, a table of logs to 110 decimal places was published under the auspices of the Royal Society.

The purpose of this essay is to explain Napier's discovery and in the process answer the question of the title. The writer is motivated in part by the fact that historical accounts are either sketchy or inaccurate or both. We shall refer to some of these at the appropriate place in the narrative. Napier's ideas are, as we shall see, quite subtle and many writers have failed to appreciate their brilliance and depth.

Napier was born at Merchiston near Edinburgh in 1550 and died in 1617. It is worth noting that Descartes lived from 1596 to 1650 while Newton lived from 1642 to 1727, the Principia having been published in 1687. Thus the mathematical tools available to Napier were decidedly limited. We should stipulate that the laws of exponents were, by then, well understood. Napier's ideas exhibited a remarkably clear conception of the logarithmic function, the term "logarithm" having been coined by Napier himself. This was at a time when the concept of function was only vaguely understood by the scientific community of his day.

Moreover, he perfected the notation for the decimal representation of numbers, his notation being essentially that in use today. The decimal representation of numbers had been earlier described by S. Stevin, who built upon earlier work, and whose notation was not as elegant as Napier's.

Two books were published on logarithms. The first in 1614 was titled "MIRIFICI LOGARITHMORUM CANONIS DESCRIPTIO" which has been translated as "Description of the wonderful canon of logarithms." This contains a table of logarithms together with rules for the solution of triangles, both plane and spherical, with the use of the "canon."

The second was published posthumously in 1619 and was titled "MIRIFICI LOGARITHMORUM CANNONIS CONSTRUCTIO" or "Construction of the wonderful canon of logarithms." It is in this work that he gives an account of the method by which the table was constructed as well as the properties of his logarithmic function, properties essential to the construction. It is this account that we propose to analyse and upon which we shall elaborate.

Before proceeding, we should add parenthetically that Napier was well-known and highly esteemed in theological circles for his analysis and interpretation of the Book of the Revelation of St. John the Divine!

§2. THE PROBLEM. The end of the 16th and beginning of the 17th century was a period of profound astronomical research with such celebrated scholars as J. Kepler (1571–1630), Tycho Brahe (1546–1601), Galileo Galilei (1564–1642). The need for carrying out elaborate calculations involving trigonometric functions was very pressing. It was therefore urgent that some procedure be sought to shorten the labor required to perform these calculations. One such aid was the use of the identity $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$ which was given the tongue-twisting name of prosthaphaeresis. There is some evidence that the method was used by Brahe and his assistant Wittich to whom the method is sometimes attributed. Another was the use of identity $4\,AB = (A + B)^2 - (A - B)^2$, which is sometimes referred to as the method of "quarter squares".

Clearly these were inadequate. Ideally, what was needed was a function from $(R_+^*, \cdot\,)$ to $(R, +)$ which converted multiplication to addition, in other words, a logarithmic function.

With the laws of exponents in mind, the most obvious approach to defining such a function is to begin with a fixed real number c (which, in what follows, we take to be < 1), calculate $b_n = c^n$ $(n = 1, 2, \cdots)$ and call n the logarithm of b_n. Moreover we need only calculate c^n for $n = 1, 2, \dots, k$ where k is that value which makes c^k about $1/2$. For if $b = c^m > 1/2$, we find l so that $\alpha = b/2^l$ with $\alpha \leqq 1/2$ and then $\log b$ is determined by $\log \alpha$ and $\log 1/2$.

Reasonable though this approach may be, it is subject to difficulties which are not easily overcome and which we now describe. Moreover, we begin by imposing reasonable restrictions.

(i) The value of c should be chosen so that the calculation of $c^n (n \in \mathbb{N})$ is arithmetically simple.

(ii) The value of c^n should not decay too rapidly i.e. the values c^n and c^{n+1} should be relatively close to one another.

(iii) Given two values c^r and c^s, if a is such that, $c^r < a < c^s$, we require a method for finding α such that $a = c^\alpha$. The method should be accurate and easy to use.

(iv) The labor of carrying out the needed computations should be within manageable bounds.

Napier defines a mapping, which we describe in detail below, and is thereby led to choose $c = 1 - \lambda$ with $\lambda = 10^{-\sigma}$ and $\sigma \in \mathbb{N}$. Let us stress at once that these powers are reference points and are the basis upon which the canon is constructed. As we shall see, the choice of σ determines the degree of accuracy of the final product.

If we write $a_m = \lambda^{-1}(1-\lambda)^m$ ($m \in \mathbb{N}$), then a_m conforms to requirements (i) and (ii) for $a_m = a_{m-1} - \lambda a_{m-1}$. Since $a_{m-1}\lambda$ is merely a shift of decimal, the value of a_m is easily obtained from a_{m-1} by a simple subtraction. The arithmetic could hardly be simpler. Moreover, with a proper choice of σ, a_m can be made to decay as slowly as we please.

Let $b_n = c^n$, and suppose we take $\log b_n = n$. If then a is such that $c^{n+1} < a < c^n$, and we find $\log a$ by linear interpolation, the error E satisfies

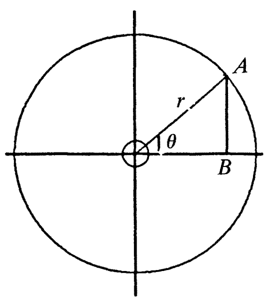
$$|E| \leqq \frac{1}{2}10^{-5} \text{ and is relatively sharp.}$$

Suppose c^k is about $1/2$ with $c = \lambda(1 - 10^{-5})$, then k is about $7 \times 10^{\sigma}$. If therefore we aim for 7 figures of accuracy, we must choose $\sigma = 7$ thus necessitating initially 7,000,000 calculations. This is an overpowering amount of labor. In addition, we have the labor of interpolation.

We state at once that this is not the method adopted by Napier.

It seems to this writer virtually certain that as Napier embarked on his project, he soon perceived the shortcomings of such a comparatively straightforward approach. He therefore sought and discovered an alternate route, deeper, more effective and ultimately successful. There is evidence that Napier started to think about the problem around 1594 but there is no record of any of his false starts. We should remark that a Swiss contemporary named JOST BURGI used this straightforward approach and published a table in 1620 well after Napier's work had been recognized and widely appreciated. Burgi's table proved to be of very limited use.

§3. PRELIMINARY REMARKS. Napier's motivation was the simplification of calculations related to the solutions of triangles, especially spherical triangles which were crucial in astronomy. In Napier's day and indeed for some time thereafter, the sine of an angle was not viewed as a ratio. It was taken to be the leg of a right triangle. More specifically, suppose we have a circle of radius r and an angle θ.



Then sine of θ was taken to be AB. The fact that the sine changed with the radius was not a serious impediment.

Napier sets out to construct a table which consists of the logarithms of $\sin\theta$. In this table Napier chooses $r = 10^7$ and his table consists of logarithm $10^7 \sin\theta$ ($30° \leqq \theta \leqq 90°$) and at increments of 1′. Thus, although he has in mind applications to trigonometry, the table is in reality a table of logarithms ranging from 10^7 to $10^7/2$ with increments which are not arithmetic but "geometric". Finally we note that although the words "cosine" and "tangent" had not yet come into use, his table includes logs of cosines and tangents.

Moreover, let us stress that we have used the word "logarithm" generically. Napier's logarithms, which we shall denote below by $LN(x)$, have somewhat different properties from the standard function $\log x$, which as we know, defines an isomorphism of $(R_+^*, \cdot)$ and $(R, +)$. These differences are not significant but have led to misinterpretations by some historians. Even the redoubtable French general has implied mistakenly, that Napier's function was an isomorphism from $(R_+^*, \cdot)$ to $(R, +)$. This assumption is also inherent in other authors' assertion that Napier chose the base e or the base $1/e$. Though not an isomorphism, the fundamental idea, however, of accurately converting multiplication to addition is essentially preserved.

§4. THE CONSTRUCTION. To determine a correspondence between a set in "geometric progression" and one in "arithmetic progression", that is, an exponential mapping which is the key element in defining a logarithmic function, Napier ingeniously resorts to a model from mechanics. It is based on the simple idea that the displacement of a point which moves with constant velocity is "arithmetic" while the displacement of a point which moves with a velocity proportional to the displacement is "geometric". The correspondence between two such points defines the required mapping. Here is Napier's model. Let TS be a segment of fixed length

$$w = 10^7.$$

The choice of the fixed length TS is motivated by the fact that Napier was interested in logarithms of $\sin\theta$ and is not a whimsical one! Given his objective, the choice was perfectly reasonable, the value for w being dictated by the degree of precision desired.

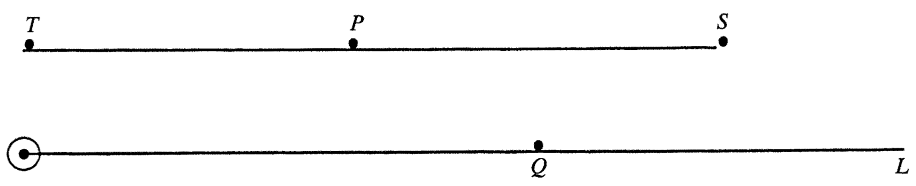


Figure 1

Let OL be another line extending to infinity. At $t = 0$, let the points start at T and O respectively. The point Q moves on OL with constant velocity v_0. The point P moves on TS in such a way that its velocity is proportional to the distance PS and we assume its velocity at T to be v_0. The velocity of P therefore, decreases from v_0 at T to 0 at S.

If at a time t, the point P is at a distance x from S, i.e. $PS = x$ and Q at a distance y from O, i.e. $OQ = y$, then Napier defines

$$y = \text{logarithm of } x.$$

We shall write

$$y = LN(x) \tag{1}$$

to distinguish this function from the standard logarithm. We shall shortly establish the connection between the two.

Evidently

$$LN(w) = 0, \tag{2}$$

since when P is at T, we have $PS = w$ while $OQ = 0$.

Before proceeding to the table of logarithms, let us use mathematics not then available to Napier to see exactly what his function $LN(x)$ is.

We have by Napier's conditions,

$$\frac{dx}{dt} = -kx$$

and at $t = 0$, $x = w$ and $dx/dt = v_0$. Hence $k = v_0/w$.

Since

$$\frac{dy}{dt} = v_0, \quad \text{we get}$$

$$\frac{dy}{dx} = -\frac{w}{x} \quad \text{or} \quad y = -w \ln x + c$$

But when $t = 0$, $x = w$ and $y = 0$, hence $c = w \ln w$ or finally

$$LN(x) = w(\ln w - \ln x)$$
$$= \ln\left(\frac{w}{x}\right)^w.$$

Many writers correctly state this fact about Napier's logarithms but it is not very illuminating except to verify that his logarithmic function is not an isomorphism. It is moreover, disingenuous to imply that he clumsily used a complicated function when he could have used a simple one! From (6) we have

$$LN(ab) = LN(a) + LN(b) - w \ln w;$$

hence

$$LN(ab) \neq LN(a) + LN(b).$$

A simple transformation however, gives $LN(x)$ a more familiar form. Let $\bar{y} = y/w$, $\bar{x} = x/w$, then, from (6)

$$\bar{y} = -\ln \bar{x} = \log_{1/e} \bar{x}.$$

This fact has led many writers to state erroneously, that Napier chose e or $1/e$ as the base of his logarithms. Napier's function is not an isomorphism, though, as we have remarked above, closely related.

It is interesting to observe that a modification of Napier's model will lead easily to the natural logarithm. Namely, assume that the velocity of P is proportional to OP while that of Q is constant. Assume too, that at $t = 0$, $OP = 1$ and $Q = 0$. The underlying differential equation has the natural logarithm as its solution. The reader may assign this as an exercise to a class in calculus.

§5. PROPERTIES OF $LN(X)$. Napier begins by doing some geometry in order to justify the claim that as P moves geometrically, Q moves arithmetically. We interpret his reasoning as follows.

Let P_1, P_2, P_3 be points on TS in geometric progression i.e.

$$P_1S : P_2S = P_2S : P_3S \tag{1}$$

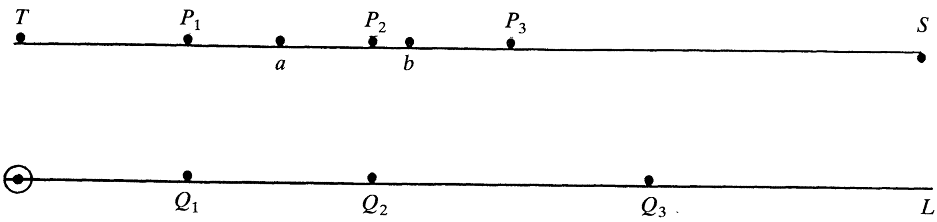


Figure 2

Let the corresponding points on OL be Q_1, Q_2, Q_3.

Let a be an arbitrary point in P_1P_2 and b the corresponding point in P_2P_3, that is

$$aP_1 \colon aP_2 = bP_2 \colon bP_3 \tag{2}$$

Let v_x be the velocity of the point P when it is at x ($= PS$). From (1) we get

$$P_1P_2 \colon P_1S = P_2P_3 \colon P_2S, \tag{3}$$

that is,

$$P_1S \colon P_2S = P_1P_2 \colon P_2P_3 = \lambda \text{ (say)}. \tag{4}$$

On the other hand, from (2) we have

$$P_1P_2 \colon aP_1 = P_1P_2 \colon P_2P_3 = \lambda \tag{5}$$

By construction, and from (4) and (5)

$$\begin{aligned} v_a \colon v_b &= aS \colon bS = P_1S - aP_1 \colon P_2S - bP_2 \\ &= \lambda P_2S - \lambda bP_2 \colon P_2S - bP_2 = \lambda. \end{aligned}$$

Since $P_1P_2 \colon P_2P_3 = \lambda$, it is very plausible to assume, and indeed Napier does assume, that the point a traverses the segment P_1P_2 in the same time that b traverses P_2P_3.

Since the velocity on OL is constant, and we have shown that the times to traverse P_1P_2 and P_2P_3 are the same, it follows that

$$OQ_2 - OQ_1 = OQ_3 - OQ_2.$$

This conclusion forms the basis for Napier's further developments. The reader will note that Napier has, in effect, integrated the underlying differential equation.

We use the above analysis to derive properties of $LN(x)$, properties essential in the construction of the table of logarithms.

Theorem 5.1. *If* $x_1x_4 = x_2x_3$, *then*

$$LN(x_1) + LN(x_4) = LN(x_2) + LN(x_3) \tag{6}$$

Proof: Referring to figure 2, let

$$P_1S = x_1,\ P_2S = x_2,\ P_3S = x_3 \text{ with } x_1 \colon x_2 = x_2 \colon x_3,$$

and let $OQ_1 = y_1$, $OQ_2 = y_2$, $OQ_3 = y_3$, then

$$y_i = LN(x_i) \qquad (i = 1, 2, 3),$$

and since $y_2 - y_1 = y_3 - y_2$, we get

$$LN(x_2) - LN(x_1) = LN(x_3) - LN(x_2). \tag{7}$$

Now choose x_4 so that

$$x_2 : x_3 = x_3 : x_4,$$

then $x_1 : x_2 = x_3 : x_4$ and

$$LN(x_3) - LN(x_2) = LN(x_4) - LN(x_3). \tag{8}$$

Combining (7) and (8) we get the result. Dropping the subscripts, we have that if $ab = cd$, then $LN(a) + LN(b) = LN(c) + LN(d)$.

Although $LN(x)$ does not satisfy the additive condition, it does however, satisfy the following modified property:

Theorem 5.2. *If TS is denoted by w, then*

$$LN(wab) = LN(wa) + LN(wb)$$

Proof: Using theorem 1, we have

$$LN(wab) + LN(1) = LN(wa) + LN(b)$$

and

$$LN(wb) + LN(1) = LN(w) + LN(b)$$

Since $LN(w) = 0$ the result follows.

Cor.

$$LN(wc^n) = nLN(wc)$$

for $n = 0, 1, 2, \ldots$.

The proof is a straightforward induction.

The next step in the construction is to find bounds for $LN(x)$. These bounds are absolutely crucial to the calculation of the table of logarithms.

Theorem 5.3. *We have the following inequalities*:

$$x\left(\frac{w}{x} - 1\right) < LN(x) < w\left(\frac{w}{x} - 1\right) \tag{9}$$

Proof: Referring to fig. 3, if P and Q are corresponding points, then because P is slowing down from its initial velocity v_0 at T, and Q is moving with constant velocity v_0, it follows that $OQ > TP$. Hence

$$y = LN(x) > TP = TS - PS = w - x.$$

On the other hand imagine that the point P goes to P', a point to the left of T and let Q' be the corresponding point on OL, then the velocity at P' is greater than v_0, and hence

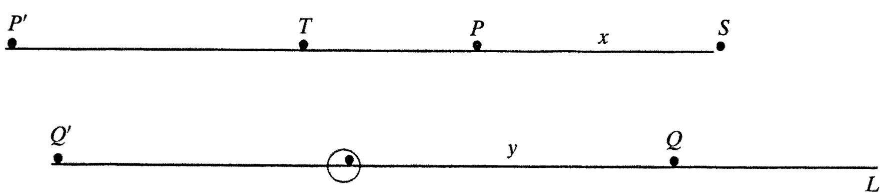


Figure 3

$$OQ' < TP'.$$

But from the geometry, we know that if $OQ' > OQ$, then

$$TS : PS = TP' : TP,$$

and hence

$$TP' = \frac{w}{x}(w - x).$$

Consequently

$$LN(x) = OQ = Q'O < TP' = \frac{w}{x}(w - x)$$

as required.

Cor. *If $a < b$, then*

$$\frac{w}{b}(b - a) < LN(a) - LN(b) < \frac{w}{a}(b - a). \tag{10}$$

Proof: Choose c so that $bc = aw$, then

$$LN(b) + LN(c) = LN(a).$$

Now apply the theorem to $c = aw/b$ to get the stated conclusion.

The reader will note once again Napier's remarkable insight since (10) gives the inequalities which follow from the mean-value theorem for $LN(x)$.

§6. THE CANON. The construction of the table takes place in four stages. The first is the calculation of a number of reference points. The second is the evaluation of the logarithms of these reference points. The third is the calculation of logarithms of intermediate values and finally the fourth step is the determination of logarithms lying outside the table.

Step 1. The reference points. These are the points

$$w(1 - c)^n$$

To understand the choice of these points, it suffices to take short time intervals and assume that the velocity of P in fig. 1 is constant in that interval. If Q_n is the position of the point Q which is n units from 0, and P_n the corresponding position of P, then it is readily seen that an approximation to P_n is given by $w(1 - c)^n$. Thus while $LN(P_n) = n$, we stress emphatically that $LN(w(1 - c)^n) \neq n$. The assertion that $LN(w(1 - c)^n) = n$ is one of the most flagrant errors made by commentators.

We have already observed that the calculation of the values

$$A_n = w(1 - 10^{-k})^n$$

for a fixed k is arithmetically comparatively simple and this fact reinforces Napier's decision to choose these points.

We have seen that in order for $w(1 - c)^n$ to be approximately $w/2$, n must be about 7,000,000 (with $c = 10^{-7}$). This is an overwhelming amount of calculation. Since, as we now know,

$$LN'(x) = -\frac{w}{x}, \tag{1}$$

and in the range $1/2 \leqq x \leqq 1$, $LN'(x)$ varies from $= w$ to $-2w$, Napier feels justified in taking larger "bites" in getting to $w/2$. This tactic, explained below, enables him to get to $w/2$ in 1600 steps and, as it turns out, retain seven decimal places of accuracy for these reference points.

Here then are the steps taken by Napier. He calculates 3 tables as follows.

Table I. Calculate $a_n = w(1-c)^n$ for $n = 0, 1, 2, \ldots, 100$. The last entry in this table is $w(1-10^{-7})^{100}$ and this is approximately $w(1-10^{-5})$. The reader should recall that if a_n has been calculated then $a_{n+1} = a_n - 10^{-7}a_n$. The second term is merely a shift of decimal. The same remark applies to the remaining tables.

Table II. Calculate $b_n = w(1-1/10^5)^n (n = 0, \ldots, 50)$. The last entry in this table is approximately $w(1-1/2000)$.

Despite the simplicity of calculating the values b_n, Napier makes an arithmetic error which affects the accuracy of subsequent calculations. We shall not dwell on this point since it has no bearing on the validity of his method.

Table III. This is a double array

$$c_{m,n} = w\left(1 - \frac{1}{2000}\right)^{m-1}\left(1 - \frac{1}{100}\right)^{n-1}$$

for $1 \leqq m \leqq 21$, $1 \leqq n \leqq 69$.

The array has 69 columns and 20 rows. The first column begins at approximately the point where Table II ended and since $(1-1/2000)^{20}$ is approximately $(1-1/100)$, the last entry of any column is approximately the second entry of the next column.

The last entry in the last column is

$$w\left(1 - \frac{1}{2000}\right)^{20}\left(1 - \frac{1}{100}\right)^{68},$$

and this is about

$$a = w\left(1 - \frac{1}{100}\right)^{69}$$

or about $w/2$.

The computation of the reference points of Tables I, II and III involves 1600 calculations, a far cry from 7,000,000.

Step 2. The Radical Table.

Having calculated the reference points, Napier now proceeds to construct what he calls the Radical Table, that is to say the logarithms of all the reference points of table III. He does this in a systematic way with the help of tables I and II.

First we calculate the logarithms of entries of tables I and II. Let us begin with table I. Using the inequalities of theorem 3, we find

$$1 < LN(w(1-c)) < 1.0000001,$$

Napier takes the value to be the arithmetic mean i.e.

$$LN(w(1-c)) = 1.00000005 \tag{2}$$

which is accurate to 14 decimal places.

If we systematically calculate $w(1-c)^k$ $(k \leqq 100)$ recursively we find $w(1-c)^{100} = 9999900.0004950$.

Using the Corollary of Theorem 5.2, we get $LN(w(1-c)^k) = kLN(w(1-c))$ and in particular

$$LN\left(w(1-c)^{100}\right) = 100.000005 \tag{3}$$

We have lost 2 decimal places in the process. Thus we have all the values of Napier's logarithms for the points in table I.

Now we calculate the logarithm of the second entry of table II viz

$$b = w(1 - 10^{-5}) = 9999900.$$

This is based on the fact that b is approximately equal to

$$a = w(1-c)^{100} = 9999900.0004950, \text{ whose logarithm we determined in (3).}$$

Writing

$$LN(b) = LN(a) + (LN(b) - LN(a)),$$

Napier estimates $LN(b) - LN(a)$ using the inequalities of Theorem 5.3. To 10 decimal places, we find

$$LN(b) - LN(a) = .0004950,$$

and therefore we evaluate

$$LN(b) = 100.0005.$$

The logarithms of all entries in table 2 are now obtained using the Corollary of Theorem 5.2, i.e. $LN(w(1-10^{-5})^k) = kLN(w(1-10^{-5}))$.

We pass to Table III. The last entry of Table II is

$$y = w(1-10^{-5})^{50} = 9995001.224804023027881. \tag{4}$$

The second entry of Table III (the first is 1) is

$$x = w\left(1 - \frac{1}{2000}\right) = 9995000 \tag{5}$$

which is approximately $w(1-10^{-5})^{50}$.

To find its log, we could proceed as above, but this would result in a significant loss of accuracy. So Napier introduces a subtle idea which we describe. Because

$$LN(x)' = -w/x,$$

the accuracy of the inequalities of Theorem 5.3 is much greater the closer x is to w. Napier uses this observation (which he has evidently discerned) as follows:

Assuming $LN(y)$ is known find the value of $LN(x)$. First choose z satisfying

$$\frac{w}{z} = \frac{y}{x}. \tag{6}$$

It is easily seen that z lies in the range of table I. Since from (3)

$$LN(z) = LN(x) - LN(y),$$

writing

$$LN(x) = LN(y) + LN(z),$$

reduces the computation to $LN(z)$. Let us call this the "method of transfer". To illustrate, let us show how to calculate $LN(x)$ of equation (5) given $LN(y)$ of equation (4). From (6), we have,

$$z = 9999998.77458344.$$

The value in Table I which is closest to z is $a = 9999999$ whose logarithm we found to be $LN(a) = 1.00000005$. Using the inequalities of Theorem 5.3 applied to a and z, we find

$$LN(x) = 5001.2504168229,$$

accurate to an astonishing 10 places of decimals.

Thus using the Corollary of Theorem 5.2, the logs of column 1 of Table III are immediately evaluated. We use the method of transfer on the last entry of column i to the second entry of column $i + 1$. The remaining logs are evaluated using the corollary of Theorem 5.2. These logarithms are accurate to 7 places of decimals.

Step 3. This step consists of interpolating at intervals of 1′. This is done using the inequalities of Theorem 5.3. In addition there is a labor saving device using the identity

$$LN\left(\frac{w}{2}\right) + LN(w \sin 2\theta) = LN(w \sin \theta) + LN\left(w \sin\left(\frac{\pi}{2} - \theta\right)\right),$$

which enables us to read off the values for $30° < \theta \leqq 45°$ from those already calculated.

In §7, we shall comment further on the accuracy of all the entries.

Step 4. The Short Table.

The final step is to find logarithms of numbers not in the range $[w/2, w]$. To do this Napier constructs what he calls the "short table". It consists of the values

$$I(A) = -LN(Aa) + LN(a)$$

for

$$A = 2^p \times 10^q \qquad 0 \leqq p \leqq 3, \qquad 0 \leqq q \leqq 7.$$

$I(A)$ does not depend upon a as is easily seen.
Suppose for example that $0 < a < w/2$.

Choose m so that

$$\frac{w}{2} \leqq 2^m a < w.$$

Then $2^m a$ lies in the range of the radical table, and

$$\begin{aligned} LN(a) &= LN(2^m a) - LN(2^m a) + LN(a) \\ &= LN(2^m a) + I(2^m). \end{aligned}$$

Thus knowing $I(2^m)$ permits us to evaluate $LN(a)$.
To find $I(A)$, we begin with

$$wa = \frac{w}{2}(2a).$$

Then by Theorem 5.1,

$$LN(a) = LN(2a) + LN\left(\frac{w}{2}\right)$$

(recall that $LN(w) = 0$).

From the radical table however, we find

$$LN\left(\frac{w}{2}\right) = 6931469.22.$$

This gives $I(2)$. By induction we get $I(2^k)$. From the relation $2^3aw = (8w/10)10a$, we find $LN(a) - LN(10a) = I(10) = 23{,}025{,}814$, and by induction $I(10^k)$, and finally $I(A)$.

A reader who has worked recently with logarithms, will not fail to recognize that

$$6{,}931{,}469 = 10^7 \ln 2$$

while

$$23{,}025{,}814 = 10^7 \ln 10.$$

Finally therefore, all information is available to calculate $LN(a)$ for any value of a.

Let us finally try to summarize Napier's method. Using a model based on mechanics, Napier defines a function from $A = [0, w]$ ($w = 10^7$), to $\mathbb{R}$. Identities satisfied by $LN(x)$ are proved and in effect the Mean Value Theorem for $LN(x)$ is used to derive inequalities satisfied by $LN(x)$. These are logarithmic-like properties.

To evaluate $LN(x)$ a subset $S \subset B = [w/2, w]$ consisting of about 1600 reference points is chosen. This subset is not random but is generated by powers of a number $c < 1$ so that control over the set is maintained. Using properties of $LN(x)$, the values $LN(x)$ for $x \in S$ are calculated. Using a subtle interpolation scheme, $LN(x)$ for $x \in B$ is calculated at intervals of $1'$. Finally a table is given to facilitate the calculation of values of x which lie outside of B.

§7. BRIEF ERROR ANALYSIS. How accurate are Napier's logarithms? It is more pertinent to ask how accurate his method is for, as we have observed, he made an error in table II which affected subsequent calculations.

We begin by estimating the error of the value of $LN(x)$ assumed by Napier. Recall that $LN(x)$ is a decreasing function. Consider two values $b > a > 0$. To calculate $LN(b)$ given $LN(a)$, we write

$$LN(b) = LN(a) - D,$$

where

$$D = LN(a) - LN(b).$$

From Napier's inequalities of Theorem 5.2, we have

$$\frac{w}{b}(b - a) < D < \frac{w}{a}(b - a).$$

Napier takes the mean of the bounds

$$A = w/2(b - a)\left(\frac{1}{b} + \frac{1}{a}\right).$$

(In fact he often divides not by a or b but by some intermediate values to render the arithmetic easier.)

On the other hand, using Taylor's theorem we find that the difference is

$$A - D = E = \frac{w(b-a)}{2}\left(\frac{1}{a} + \frac{1}{b}\right) - \frac{w}{a}(b-a)$$
$$+ \frac{w(b-a)^2}{2a^2} - \frac{w(b-a)^3}{3a^3} + \cdots .$$

A little calculation shows that

$$E = \frac{w(b-a)^3}{6a^3} + \frac{w(b-a)^4}{a^3}\left(\frac{1}{4a} - \frac{1}{2b}\right) + 0\left(\frac{(b-a)^5 w}{a^4}\right),$$

from which we may verify that the value

$$LN(9{,}999{,}999) = 1.00000005$$

is indeed accurate to 14 places of decimals. A simple further analysis confirms that the reference points are accurate to 7 places of decimals.

For the interpolated values, the error E is bounded by

$$|E| < 10^{-4}.$$

Thus the interpolated values are accurate to 4 places of decimals. However in the neighborhood of $a = 10^7 \sin\theta$ with θ close to $90°$, the accuracy is far better—we saw that above.

We could improve the accuracy by the method of transfer which entails considerably more labor. Napier, however, is content to settle for the accuracy he has achieved.

In fact, if we use the method of transfer to calculate $LN(5{,}000{,}000)$, we find

$$LN\left(\frac{w}{2}\right) = LN(5{,}000{,}000) = LN(10^7 \sin 30°) = 6931471.8055994$$

accurate to 7 decimal places. Note that $w/2$ is not one of the reference points.

The relative error of Napier's method is 10^{-10}.

§8. CONCLUDING REMARKS. Not surprisingly, the canon was greeted with great enthusiasm especially by Kepler who had been laboriously making calculations in connection with his laws.

It has the drawback we mentioned that $LN(1) \neq 0$. This drawback is an impediment to the ease of calculations but not a serious one. This weakness was recognized by Napier and during a visit by John Briggs to Napier, they discussed the possibility of constructing a table in which $\log 1 = 0$. This was subsequently completed by Briggs since Napier died soon after the visit.

The first table of hyperbolic logarithms, i.e. to base e, was first published by John Speidell in 1619. It was derived directly from Napier's table.

The odyssey of log tables is an interesting one but we shall not add any further details here.

While it is true that Napier's table was quickly overshadowed by others which were easier to use, it is well to bear in mind that it was Napier who, alone, led the way for others to follow. John Briggs' praise of Napier is one witness to this fact.

Finally, however, we cannot resist the temptation of quoting Napier's advice on forming a logarithmic table.

"Prepare forty-five pages, somewhat long in shape, so that besides the margins at the top and bottom, they may hold sixty lines of figures. Then divide each page.... Next write on the first page at the top, to the left, over the first three columns, '0 degrees'; and at the bottom...".

Alas such charm has virtually vanished from the pages of our journals.

BIBLIOGRAPHY. The entire essay is based on "The Construction of the Wonderful Canon of logarithms and their relations to their own natural numbers."

I have used the translation of William Rae MacDonald first published in 1889 and reprinted in 1966 for Dawson's of Pall Mall, London. The translation also contains a catalogue of Napier's works.

Since writing this article the author has found a reference to a lecture written by E. W. Hobson entitled "John Napier and the invention of logarithms, 1614. Cambridge Univ. Press 1914. It is to be highly recommended.

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The Mathematical Association of America wishes again to call the attention of all its members to the working arrangement between the Association and the *Annals of Mathematics* by which, in return for a certain subsidy contribution from the Association, the *Annals* has extended the size of its volume to include approximately one hundred pages of expositional articles and at the same time has made the special subscription rate to individual members of the Association of one half the regular price. A goodly number of Association members have already taken advantage of this reduced rate, but it is felt that a much larger number would probably do so if their attention were sufficiently arrested.

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