NAPIER'S LOGARITHMS AS HE DEVELOPED THEM

By W. D. CAIRNS, Oberlin College

In 1614 Napier published his *Mirifici Logarithmorum Canonis Descriptio* which incorporates his tables and in 1619 his *Mirifici Logarithmorum Canonis Constructio* which explains the method by which he formed his tables. Various editions and translations from the Latin originals are to be found, such as the combined edition of 1620, Edward Wright's 1616 translation of the *Descriptio*, the 1807 re-printing of the 1614 *Descriptio* by Maseres in the sixth volume of his *Scriptores Logarithmici*, and the 1889 translation of the *Constructio* by W. R. Macdonald. Reference may also be made to the facsimile reproduction of the title page and the first eleven pages of Wright's 1616 translation of the *Descriptio* in the Napier Tercentenary Memorial Volume (Longmans, Green and Co., 1915). Yet numerous colleges and universities have not been able to obtain material on Napier's logarithms suitable for their purposes; and it is the purpose of this paper to give the essentials of his treatment. The details of his computations throw an interesting light on the status of mathematical practice at the beginning of the seventeenth century. We leave it to the historians, however, to determine how much of this is due to Napier in particular.

Noting that he purposes to avoid all multiplications, divisions, and the more difficult extractions of roots and that his method depends on a correspondence between an arithmetic and a geometric progression, he proceeds to compute the latter as simply as possible, viz., by subtracting successively a simple fractional part. Thus, using 10^7 (in modern notation) as "radius," he takes the ratio in his "First Table" as .9999999, and one ten-millionth is subtracted successively one hundred times; the last number in this table, to the seven decimal places which he preserved, is 9999900.0004950. Replacing this last number by 9999900, the "Second Table" begins with 10^7, uses the ratio .99999 by subtracting one hundred-thousandth fifty times in succession; the last number in the table is 9995001.222927, a result slightly incorrect, for the decimal part should be .224804, as noted by Macdonald. Again, using 9995000 as "sufficiently near to" this last number, the "Third Table" begins with 10^7, uses the ratio .9995 by subtracting one two-thousandth twenty times in succession, the last number being 9900473.57808; this set of 21 numbers forms the first of 69 columns in this third table. The "easiest" number nearest to the last number, viz., 9900000, is taken as the first number in the second column, and in accordance with this, the first numbers in the 69 columns are formed, beginning with 10^7, by subtracting one hundredth 68 times in succession, that at the head of the 69th column being recorded by Napier as 5048858.8900. Next he fills in the other numbers in the 69 columns, row after row, by reducing those in the first column by one-hundredth successively; for example, the last (twenty-first) row is

9900473.5780, 9801468.8423, 9703454.1539, · · · , 4998609.4034.

This last number, he notes, is practically half the radius, and he has thus computed a whole series of geometrical means between the radius and the half radius with the ratio .9995, and can insert 49 means between each pair of consecutive numbers by the numbers of the second table used as multipliers, and can further insert 99 means between the resulting numbers by means of the numbers of the first table. His choice of the base 10^7 is now justified by the fact that the numbers thus obtained differ consecutively by no more than unity and that the decimal parts may be neglected.

At this point in Napier's explanations he introduces his theory of logarithms, centering in the definition quoted here from the *Constructio*; the numbers of the sections are given for convenience in reference:

"26. The logarithm of a given sine is that number which has increased arithmetically with the same velocity throughout as that with which radius began to decrease geometrically, and in the same time as radius has decreased to the given sine." The reader must recognize that to Napier the sine was a line, or the number measuring the line, as in the present-day line representation of functions of angles. Various theorems follow which are needed for the development of his tables, several of these being given here in substance.

27. Zero is the logarithm of the radius.

29. An upper and lower limit are derived from the definition in section 26. " · · · the given sine being subtracted from radius the less limit remains, and radius being multiplied into the less limit and the product divided by the given sine, the greater limit is produced · · · ."

33. By section 29 the first "proportional" of the first table, 9999999, has its logarithm between the limits 1.0000001 and 1.0000000, that of the second proportional, 9999998.0000001, between 2.0000002 and 2.0000000, etc. Hence these logarithms may be taken as 1.00000005, 2.00000010, etc.

36. "The logarithms of similarly proportioned sines are equi-different." This is proved by the definition of logarithms, also by a comparison of the upper and lower limits.

41. The method for finding the logarithms of sines or "natural numbers" not found in the first table will be shown clearly enough by his examples. Let the given sine be 9999975.5000000; the nearest sine in the table is the twenty-sixth number, 9999975.0000300, and is smaller than the given sine. By section 33 the limits of the logarithm of this last are 25.0000025 and 25.0000000. By the same method used in proving the rule in section 29, he proved that the difference of the logarithms of these two sines has the limits .49997122 and .49997124, whence to seven decimal places the required limits of the logarithm

are 24.5000313 and 24.5000288, and he uses 24.5000300. Similarly the limits for 9999900.0004950, the last sine in the first table, which are 100.0000100 and 100.0000000, yield 100.0005050 and 100.0004950 as the limits for the logarithm of 9999900, the second number in the second table, and from this the logarithms for all the other numbers in the second table are found by the rule for proportionals.

43. For the method of finding the logarithms of sines not found in the second table, an example will suffice. Let the given sine be 9995000, the second number of the first column in the third table. The nearest sine in the second table is the last, 9995001.222927, the limits for its logarithm being 5000.0252500 and 5000.0247500. He finds "a fourth proportional, which shall be to radius as the less of the given and table sines is to the greater." This he does "by multiplying the difference of the sines into radius, dividing this product by the greater sine, and subtracting the quotient from radius." By section 36 the logarithm of this fourth proportional differs from that of the radius by as much as the logarithms of the given and table sines differ from each other. He then finds the limits of the logarithm of the fourth proportional by aid of the first table, and adds them to or subtracts them from the limits of the logarithm of the table sine, to obtain the limits of the logarithm of the given sine. In this example, the fourth proportional is 9999998.7764614, and the limits of its logarithm, found by 41, are 1.2235387 and 1.2235386. Adding these to the former limits, he has 5001.2487888 and 5001.2482886 as the limits of the required logarithm, which is therefore taken (midway) between these, 5001.2485387.

44. From this the logarithms of all the other numbers in the first column of the third table are found by the rule for proportionals.

45, 46. It only remains to find by the same method the logarithm of 9900000, the first number in the second column, viz., 100503.3210291, after which the logarithms of all the numbers of the third table are found either by adding this last logarithm successively along any row, beginning with the first column, or by adding 5001.2485387 successively down each column. The logarithm of the last entry, 4998609.4034, is thus found to be 6934250.8007528.

47. The "Radical Table," next formed by Napier, merely enters all the numbers of the third table and their logarithms in 69 double columns, the "natural numbers" or sines to four decimal places, the logarithms to one; and from this he compiled his logarithm tables. For example, in getting the logarithm of 7489557, the nearest table sine being 7490786.6119, he finds the difference of these two to be 1229.6119. Multiply this by the radius and divide by "the easiest number," which may be either of the foregoing or the simpler 7490000, the result being 1640.1; this added to 2889111.7, the logarithm of the table sine, gives 2890752. Similarly for the logarithms of all sines within the limits of the radical table.

51, 52. By this same rule the logarithm of 5000000 is 6931469.22, of 8000000 is 2231434.68, and by a further use of the same rule the logarithm of 1000000 is 23025842.34.

54. These particular logarithms and the rule for proportionals give him the logarithms of all sines outside the table limits. For example, the logarithm of 378064, which is 1/20 of 7561280, a number in the limits of the table, is 32752756, the sum of 2795444.9 and 29957311.56, the logarithms of 7561280 and 20.

59. Napier's table is constructed in quite the same form as used at present, except that the second (sixth) column gives sines for the number of degrees indicated at the top (bottom) and of minutes in the first (seventh) column, the third (fifth) column gives the corresponding logarithm and the fourth column gives the "differentiae" between the logarithms in the third and fifth columns, these being therefore essentially logarithmic tangents or co-tangents. Reproductions of some of these pages may be seen in Macdonald's translation and in Cajori's History of Mathematics.

Suffice it to say, in closing this note, that the second book of the *Descriptio*, "On the remarkable advantage (praeclaro usu) in trigonometry of the wonderful canon of logarithms," shows the solutions of a right triangle by logarithms in the manner familiar to us, the solutions of oblique triangles by the law of sines and of tangents, and of the case where three sides are given, the last by the expedient of drawing the altitude from the intersection of sides a and c and using an auxiliary isosceles triangle of which this is the altitude and side c is one of the equal sides; then follows the portion devoted to spherical triangles, wherein Napier uses his "circulares," known to us as Napier's circular parts, to solve quadrantal spherical triangles, and completes the theory of spherical triangles essentially in the way adopted at the present day.

ON THE ORIGIN OF THE TERM "ROOT." SECOND ARTICLE

By SOLOMON GANDZ, The Rabbi Isaac Elchanan Theological Seminary, New York City

In a recent number of this Monthly[1] the writer suggested that the term "root" goes back to the Arabic *jadhr*, a word originally meaning a concrete number designating a geometric magnitude, as contrasted with an abstract number. He further referred to some passages of the algebra of al-Khowârismî which clearly defined *jadhr* as the side of a square multiplied into a square unit, and proposed that *jadhr* should not be translated as "root" but rather as "basis" or "foundation."

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[1] Vol. 33, pp. 261–265.