The Birth of Logarithms

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The BIRTH *of* LOGARITHMS

by Jack Oliver

Introduction

Some while ago, I gave a presentation on two or three occasions to the 'University of the Third Age' with the title 'History of Calculation'. At a point in the development, I talked about Napier (1550–1617) and the invention of logarithms. I also used a rather quickly made paper slide rule (about three metres long) which proved very successful. A question that was asked by more than one person was 'How were the logarithms of numbers calculated?' I must admit that I had only vague ideas of how the first tables of logarithms were constructed and promised myself that I would investigate further and possibly write a short paper on my findings. Well, that was some time ago and my investigations over this time led me from one mathematical topic to another and from one reference to another (not always giving consistent information).

Napier

I shall start with Napier and his time, backtrack to very early times before I return to Napier to fill in some extra details.

John Napier was born in Merchiston Castle near Edinburgh, Scotland in 1550. He was the Eighth Laird of Merchiston and so was not without means. He was educated at St Andrews University, Edinburgh. He then travelled through Europe whilst still young before returning home to Merchiston where he spent the rest of his life, dying in 1617. From this we can see that during his earlier years he was exposed to the learning of his day which served him well for his investigations in later life.

Napier died 25 years before Newton was born and so didn't have the benefit of the tools of calculus. At this time, working with fractional quantities was very cumbersome and so mathematicians tried to work with whole numbers as far as possible. Napier himself invented the 'decimal point' which was to be of help to him in constructing his logarithmic tables. It must be said that the time was ripe for such an invention; in 1585 Simon Stevin had designed a precursor of decimal notation.

Napier was a man with a curiosity. He designed mechanical artefacts (that weren't made), wrote early Science Fiction and wrote religious tracts among other things. The interests for which he is best remembered are 'Napier's Bones or Rods' which were devices that made multiplication and division easier and his making of a table of logarithms. From this we can see that he was keenly interested in ways that the arithmetical operations of multiplication and division might be simplified.

Before Napier

At this time the scientific world was greatly interested in observational astronomy of which there was a long tradition going right back to the times of Ancient Egypt and the peoples of Ancient Mesopotamia. An important tool of the astronomer was a table of chords of a circle. Such a table gives the lengths of the chords corresponding to the angles subtended by those chords at the centre of the circle or the corresponding half-chords and half-angles. The earliest known table is on a Babylonian cuneiform tablet that dates back to the earlier half of the second millennium BC and lists the 'secants' of 15 angles between 45° and 30°.

From then until Napier's time there was a continued improvement of both the completeness and accuracy of such tables. Of particular note is a table of Claudius Ptolemy's (of *Almagest* fame) (*ca.* 85–165) which lists central angles from 0° to 180° in steps of $\frac{1}{2}°$ and their chord lengths. A table of chords is attributed to Hipparchus who flourished about 250 years before Ptolemy and this may have been the inspiration for the latter's table. However, the table of Hipparchus has been lost. Continued improvements by Arabic and European mathematicians had by the time of Napier produced tables to 10 significant figures for every 10″ of arc.

Chord Tables

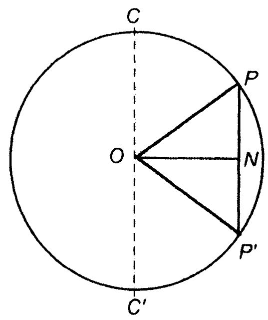

Fig. 1 We can create tables of chords or half-chords corresponding to the angles or half-angles that these chords subtend at the centre of the circle

Table 1a

Angle POP'	Chord PP'
0	0
...	...
180	CC'

or

Table 1b

Angle PON	Half-chord NP
0	0
...	...
90	OC

These tables are very much like tables of sines, particularly Table 1b. Originally the construction of such tables required a great amount of work involving the geometry of angles. Today we can construct such tables very easily using a spreadsheet, for example. Of course, that involves a lot of hidden mathematics; we are standing on the shoulders of giants who did the donkey work for us. At this point, it could be instructive to use a spreadsheet to construct the following chord table (Table 2) perhaps at intervals of 1°:

Table 2

Angle PON	Half-chord NP
90	10^7
θ	$10^7 \sin(\theta)$
...	...
0	0

We have taken the radius of the circle to be 10^7 as did Napier, and we have constructed it so that we can combine it with Napier's logarithmic table.

Although this work was developed over a long period of time, it represents a huge amount of work. What was it all for?

It is not the purpose of this paper to look at how these chord tables were constructed. Suffice it to say that they were. What we wish to look at is how starting from these tables, Napier constructed the very first logarithm tables.

As mentioned above this was a time of great work in observational astronomy and contemporaries of Napier included Tycho Brahe (1546–1601) in Denmark and Johannes Kepler (1571–1630) in Germany, among others. Their investigations led to calculations with numbers from the above-mentioned chord tables and any help that would reduce the slog of the arithmetic involved would have been greatly appreciated. Many of their calculations involved multiplications and divisions which were particularly tedious. It was Napier's inspiration to extend this chord table by adding extra columns which would reduce the operations of multiplication and division in trigonometry to addition and subtraction which are so much simpler to perform.

Napier's Work

First we shall define what Napier meant by a logarithm and we shall do this by referring to a geometrical model as he did. We shall use some modern notation rather than the terms that he used.

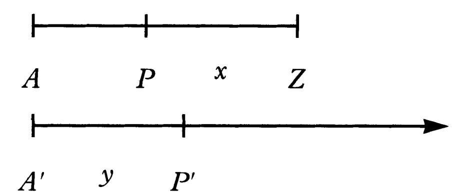

Fig. 2

As in Figure 2, take a half-line with origin A' and a line segment AZ. Now think of two particles p', p starting at the same time and moving to the right, the one at A' moving with constant speed and the other starting at A and moving according to the following rule. When particle p' has reached P' particle p has reached P such that the speed of p is proportional to the distance remaining to Z and the initial speed of p is the same as the initial speed of p', then

$$y = \text{naplog}\,(x)$$

where $x = PZ$ and $y = A'P'$ and 'naplog' refers to the logarithm of x as defined by Napier.

Notes:

1. This definition could have been simplified if Napier had had access to Cartesian coordinates and the idea of moving particles could have been avoided.
2. This is certainly not the definition of logarithms to base e that we know today although these are sometimes known as 'naperian logarithms' in honour of Napier but perhaps are better called 'natural logarithms'.
3. This is a continuous model and would not have been easy to analyse without calculus.

At this point Napier immediately moved on to a discrete model based on the above. Rather than thinking of particles moving smoothly along the above lines, Napier concentrated his attention on particular points on the two lines as in Figure 3.

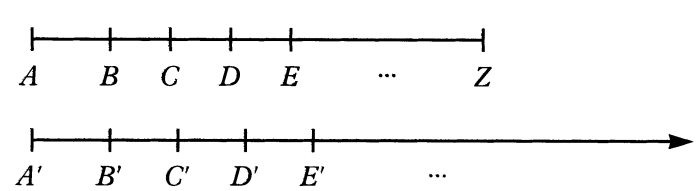

Fig. 3

On the second line A', B', C', D', E', etc., are equally spaced and are reached successively in equal 'moments' of time. The corresponding positions on the first line are A, B, C, D, E, etc. (and the points are getting closer together). It should be noted that Napier did not have access to exponential notation and numbers such as 10^7 he would write as 10000000 (without spaces). For reasons of avoiding fractions and achieving the accuracy he wanted, Napier chose the length AZ to be 10^7 units. This also was an appropriate choice based on the chord tables that were available. $A'B'$ and AB are essentially equal and on the scale chosen are equal to 1. Now from the above definition of 'naplog' and the above units of length, we can write:

$$BZ = 10^7 - 1$$
$$= 10^7(1 - 10^{-7}) \qquad \ldots \square$$

$$\frac{AB}{AZ} = \frac{BC}{BZ}$$

$$\frac{1}{10^7} = \frac{BC}{(10^7 - 1)}$$

$$BZ = 1 - 10^{-7}$$
$$CZ = BZ - BC$$
$$\ldots$$
$$CZ = 10^7(1 - 10^{-7})^2 \qquad \ldots \square$$

Similarly from

$$\frac{AB}{AZ} = \frac{CD}{CZ}$$

we can obtain

$$DZ = 10^7(1 - 10^{-7})^3 \qquad \ldots \square \quad \text{and so on.}$$

Hence we find the lengths AZ, BZ, CZ, ... are in geometrical progression with a common ratio of $(1 - 10^{-7})$ corresponding to the lengths $A'B'$, $B'C'$, $C'D'$, ... which are in arithmetic progression with a common difference of 1. Essentially we now have the tools for constructing a logarithmic table where the numbers are represented by the lengths AZ, BZ, CZ, ... and their 'naplogs' by the lengths $A'B'$, $B'C'$, $C'D'$, ... Perhaps it would be more accurate to say that we would be constructing an antilogarithm table as the numbers are 10^7, $10^7(1 - 10^{-7})$, $10^7(1 - 10^{-7})^2$, $10^7(1 - 10^{-7})^3$, ... and their corresponding 'naplogs' are 0, 1, 2, 3, ...

Napier thought that if he could calculate sufficient numbers from their 'naplogs' he could construct a table of naplogarithms by interpolations and other techniques. But how would he calculate such numbers as $10^7(1 - 10^{-7})^3$? There was a way to simplify the calculations rather than successive multiplications by $(1 - 10^{-7})$. Here is the background mathematics. The difference between two successive numbers is

$$10^7(1 - 10^{-7})^{n+1} - 10^7(1 - 10^{-7})^n = -(1 - 10^{-7})^n$$

and this is equal to the negative of $10^7(1 - 10^{-7})^n$ divided by 10^7. So Napier could calculate a number from the previous number by moving the decimal place in the previous number seven places to the left and then subtracting the result from the previous number.

Here are the first few rows of such a table:

Table 3

Number	Subtract	Number	Naplog
10 000 000	1.000 000 0	10 000 000	0
9 999 999	0.999 999 9	9 999 999	1
9 999 998.000 000 1	0.999 999 8	9 999 998	2
9 999 997.000 000 3	0.999 999 7	9 999 997	3
9 999 996.000 000 6	0.999 999 6	9 999 996	4
9 999 995.000 001 0	0.999 999 5	9 999 995	5
9 999 994.000 001 5	0.999 999 4	9 999 994	6
9 999 993.000 002 1		9 999 993	7

Notes on Table 3:

1. In the first column, 10 000 000 is the length of the radius from the chord table. Going down the table we get lengths of successive half-chords.
2. The second column gives the number to be subtracted from the number to the immediate left in the first column to give the number to the lower left in the first column. The fractional parts of the numbers in the first column being dropped in calculating the second column number.
3. The third column gives the integral part of the number in the first column which would have been that which Napier would have used.
4. The final column gives the naplog of the corresponding number in the third column.
5. The difference between the numbers in the third column from an AP would have been very little considering how close the common ratio is to 1.
6. In modern terms, naplog($10^7\sin(1')$) = 81 425 711. You might like to calculate this for yourself on a calculator using Napier's definition of a 'naplog'. At a single step at a time in constructing Table 3, this would represent a hopeless task.
7. However, it is possible to proceed at 100 or even 1000 steps at a time. Let us look at the mathematics of this as we did above for a single step. We will look at the situation for 1000 steps.

   $$10^7(1 - 10^{-7})^{n+1000} = 10^7(1 - 10^{-7})^n = \ldots$$
   $$= 10^7(1 - 10^{-7})^n(-10^{-4} + \text{terms less than } 10^{-8})$$
   $$\approx -10^{-4}\{10^7(1 - 10^{-7})^n\}$$

   In words, Napier could calculate a number from the number 1000 before it by moving the decimal place in the previous number four places to the left and then subtracting the result from the previous number.
8. The table can be extrapolated backwards for numbers greater than 10 000 000 to give negative naplogs. naplog(10 000 001) = –1, etc. to well within the accuracy for which Napier was seeking.
9. Having obtained the naplogs of a number of numbers, he could easily find the naplog of the product of any two of these numbers. This would have been an important

technique in extending the table. But see the example on products below.

10. If we look at the decimal digits to the far right of the numbers in the first column, we note that they form a sequence of triangular numbers. Perhaps this might have been used in finding numbers further down the column!

It is instructive at this point to construct a simplified table of 'naplogs' with a spreadsheet. We will choose 10^3 as the radius of our circle which leads to a common ratio of $(1 - 10^{-3})$.

Table 4

Number	Naplog
10^3	0
$10^3(1-10^{-3})$	1
$10^3(1-10^{-3})^2$	2
...	...
$10^3(1-10^{-3})^{100}$	100

The numbers in the first column of Table 4 can be calculated by using the formula $10^3(1 - 10^{-3})^n$ where n is the value of the 'naplog' in the second column and it would be appropriate to calculate it to three decimal places. Alternatively, each successive number can be found by subtracting one thousandth of its predecessor from itself as described above.

Try 'multiplying' two numbers from the left column with a calculator, add their 'naplogs' $n_1 + n_2$ from the second column and check against the result of this in the second column back into the first column. You will have to divide your result by 10^3 to adjust to the 'correct' result.

$$\begin{aligned} N_1N_2 &= 10^3(1 - 10^{-3})^{n_1}10^3(1 - 10^{-3})^{n_2} \\ &= 10^6(1 - 10^{-3})^{n_1+n_2} \\ &= 10^3 P \end{aligned}$$

You might investigate a similar procedure for division.

It is clumsy, but it works and the table is not all that appealing. Napier spent more than 20 years to 1614 performing all the calculations and interpolations and merging his results with a chord table. Be it noted that he didn't have the benefits of calculators or spreadsheets! The results of all his labours were 'Mirifici logarithmorum canonis descriptio' or, in English, 'A Description of the Wonderful Law of Logarithms', a table of logarithms with rules for their use.

From the preceding, we see that Napier's main aim was to make the operations of multiplication and division easier when applied to trigonometric functions in a chord table. He proceeded to construct a table, the outline of which is given below:

Table 5

Arches	Sines (arches)	Naplog (sines)	3rd column – 5th col	Naplog (sines)	Sines (arches)	Arches
0						90
...						...
45						45

'Arches' (1st and 7th columns) refers to half the angle subtended by the chord at the centre of the circle and the table contained entries for every successive minute of arc. The sines (arches) columns (2nd and 6th) gave the half-chord lengths corresponding to the angles of the 1st column. The naplog (sines) columns (3rd and 5th) gave the naplogs of the corresponding numbers in the 2nd and 6th columns. The 4th column gave the result of subtracting the number in the 5th column from that in the 3rd column to give the naplog of the tangent of the corresponding angle in the 1st or 7th column.

$$\begin{aligned} \text{naplog}\{\sin(\theta)\} - \text{naplog}\{\sin(90-\theta)\} &= \text{naplog}\{\sin(\theta)/\sin(90-\theta)\} \\ &= \text{naplog}\{\sin(\theta)/\cos(\theta)\} \\ &= \text{naplog}\{\tan(\theta)\} \end{aligned}$$

If we put a negative sign before a number in the 3rd or 5th columns, we obtain:

$$\begin{aligned} -\text{naplog}\{\sin(\theta)\} &= \text{naplog}\{\text{cosec}\,(\theta)\} \\ &= \text{naplog}\{\sec(90-\theta)\} \end{aligned}$$

In words, the negative of a number in the 3rd or 5th column gives the naplog of the secant of the angle in the 7th or 1st column.

So from this table we can obtain the naplogs of the sines, tangents and secants of angles and hence the sines, tangents and secants from the 2nd and 6th columns.

Further Developments

Although there is much more that could be said about the structure of Napier's table, what has been said will be sufficient for the purpose of this article.

The table was received with great enthusiasm when it was first published and in particular proved a boon for the astronomers of the time. Even 180 years later, Laplace gave it an accolade by describing it 'as doubling the life of the astronomer' because of all the time that was saved in simplifying calculations.

In 1615 the year after the first publication of the tables, Henry Briggs travelled to Scotland to pay his respects to Napier and stayed there for a month before returning to London. Briggs was then Professor of Geometry at Gresham College in London. During this month, the two agreed that there would be much benefit in reconstructing the table so that the logarithm of 10 was 1 and the logarithm of 1 was 0. Thus was born the idea of Common or Briggsian logarithms which were to be used from the 1620's to well into the late 1900's when the electronic calculator superseded logarithms for the purposes of calculation.

It must be said that the 'time was ripe' (the end of the second decade of the 1600's) for a leap forward to better methods of making calculations. Briggs had produced a table of 14-figure logarithms in 1624. In 1620 Edmund Gunter who was a colleague of Briggs had produced a table of common logarithms of sines and tangents. Also at this time the first slide rules were made which depended for their working on the theory of logarithms.

The final part of this article will be devoted to the ideas that Briggs used in constructing his table of common logarithms. Here it would be well to note the differences between Napier's original table and the table that was finally published by Briggs in 1624. Napier's table was defined by a fixed point (the naplog of 10^7 was 0) and a common ratio of $(1 - 10^{-7})$ as described above. Briggs' table had two fixed points corresponding to the numbers 1 and 10 whose Briggsian logarithms were 0 and 1. With this choice, Briggs ensured that his logarithms were ideally suited to the purposes of calculation as is testified by their longevity of use over many centuries. Briggs' table was not complicated with trigonometric considerations and so the whole concept was much simpler than that of the table devised by Napier.

And now to some discussion of the ideas behind the table constructed by Briggs.

The Calculation of Briggsian Logarithms

We shall attempt to illustrate the thinking behind the work of Briggs by first of all finding the logarithm of a single number to the base of 10 and then indicate how this could have been used as the basis to calculate a whole table of logarithms.

For our present purpose, we shall always regard $\log(x)$ as a logarithm to the base of 10 where the logarithm is the power to which 10 must be raised to give x.

We shall try to find the value of log(7). We emphasize that the only arithmetical operations that were available to Briggs were those of $+$, $-$, $\div$, x, $\sqrt{}$, and a lot of time and patience. The square root operation was particularly important as you will see.

First, we shall construct the following table:

Table 6

Reciprocal of Power	Power	Number
2	0.5	3.16227766
4	0.25	1.77827941
8	0.125	1.333521432
16	0.0625	1.154781985
32	0.03125	1.074607828
64	0.015625	1.036632928
128	0.0078125	1.018151722
256	0.00390625	1.009035045
512	0.001953125	1.004507364
1024	0.000976563	1.002251148
2048	0.000488281	1.001124941
4096	0.000244141	1.000562313

Notes on Table 6:

1. The 1st column contains powers of 2.
2. The 2nd column contains the reciprocals of those powers in decimal form. These values can be obtained by division.
3. The numbers in the 3rd column are found by the process of finding successive square roots of first 10 and then the square root of that.

   $3.162\,277\,66 = \sqrt{10}$, $1.778\,279\,41 = \sqrt{\sqrt{10}}$, etc.
4. The values are worked out to 8 decimal places. Briggs worked to many more.
5. Since $3.162\,277\,66 = 10^{1/2}$ then $\log(3.162\,277\,66) = 1/2 = 0.5$ and $1.333\,521\,43 = 10^{1/8}$ then $\log(1.333\,521\,43) = 1/8 = 0.125$

Armed with the above information, we can continue with our original problem of finding log(7).

If we put $7 = 10^x$ then we can write $7 = 10^{1/2} \times 2.213\,594\,36$ where $2.213\,594\,36 = 7/10^{1/2}$ and $10^{1/2} = 3.162\,277\,66$ was taken as the largest number in column 3 that is less than 7.

We now deal with 2.213 594 36 in the same way and find that $2.213\,594\,36 = 10^{1/4} \times 1.244\,795\,59$.

Then $1.244\,795\,59 = 10^{1/16} \times 1.077\,948\,57$ and so on. We carry this on until we have achieved the desired accuracy. Back-substituting we find

$$7 = 10^{1/2} \times 10^{1/4} \times 10^{1/16} \times 10^{1/32} \times 10^{1/1024} \times 10^{1/4096} \times \ldots$$

$$\text{when} \quad \log(7) = 1/2 + 1/4 + 1/16 + 1/32 + 1/1024 + 1/4096 + \ldots = 0.844\,970\,70$$

which is correct to about 0.015%.

Now we know how to find a logarithm. The above is essentially the process that Briggs used to calculate logarithms. Other tools that Briggs could have used were the 'Laws of Logarithms', in particular $\log(mn) = \log(m) + \log(n)$ so that having calculated log(25) and log(7) he could find log(175) as log(25) + log(7), log(7000) would be log(1000 x 7) = 3 + log(7). Even then ten years of work was needed until in 1624 he finally published his 'Arithmetica logarithmica' being a 14-place table of common logarithms of the numbers from 1 to 20 000 and from 90 000 to 100 000. The gap from 20 000 to 90 000 was filled in during the next few years.

And so here we have the story of the birth of logarithms. The motivation behind the construction of the tables above was an aid to calculating products and quotients. For Napier it was as an aid to calculations in trigonometry and hence to astronomy. It was a pity that Napier wasn't able to divorce the work of calculation from the trigonometry; he would have saved himself much work and time.

Briggs saw a simpler solution but again was concerned with easing the calculations involved in multiplication and division. To both their credits, their work was used well into the late 20th century when the electronic calculator took over.

The other face of logarithms is the background mathematical theory together with the corresponding theory of exponential functions. These provide powerful tools for studies in the physical, biological and social sciences. But that is a whole new story.

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