A Brief History of Logarithms

R. C. Pierce, Jr.

R. C. Pierce, Jr., has taught at Leesville Junior High, McNeese State University, and is currently Assistant Professor of mathematics at the University of Houston-Downtown College (formerly South Texas Junior College) where he has been since 1967. He has worked in developing the Business Mathematics curriculum at UH-DC and is the coauthor of "Elementary Calculus for Business, Economics, and Social Sciences" (with Chaney Anderson).

The concept of a logarithm made its first appearance in ancient Babylonia where baked clay tablets have been found which contain tables of successive powers of whole numbers. In some of these records the question is asked: "To what power must a certain number be raised in order to yield a given number?" In modern terminology this is equivalent to asking, "What is the logarithm of a given number to a given base?" However, the Babylonians did not appear to be interested in logarithms as a computational aid but rather as something to be used to solve certain types of problems [1, p. 32].

Archimedes made an observation that is the basis of our modern logarithms. (He defined the "order" of a number to be equivalent of the exponent where the base is 100,000,000.) Archimedes then observed that the addition of orders corresponds to finding their product, a result which we know as the first law of exponents [1, p. 139].

The development of logarithms began in earnest during the late fifteen hundreds. At this time the astronomer Tycho Brahe (1546–1601) was trying to disprove the Copernican theory of planetary motion. He was doing calculation by the method which had the rather preposterous name of prosthaphaeresis. Prosthaphaeresis (from the Greek meaning addition and subtraction) is a method of computation using trigonometric identities [1, p.339]. Following is a sample calculation of this type:

Find the product of (2250)(1219).

Using the identity

$$(\cos A)(\cos B) = \frac{\cos(A + B) + \cos(A - B)}{2},$$

we let $\cos A = .2250$, where the decimal is placed so that .2250 represents the cosine of an angle. Similarly, let $\cos B = .1219$. Therefore $A \approx 77°$ and $B \approx 83°$.

Replacing quantities by their equals yields

$$(.2250)(.1219) = \frac{\cos(77° + 83°) + \cos(77° - 83°)}{2}$$

$$= \frac{\cos(160°) + \cos(-6°)}{2}$$

$$= \frac{-.9397 + .9945}{2}$$

$$= \frac{.0548}{2}$$

$$= .0274.$$

Replacing the decimal yields

$$(2250)(1219) \approx 2{,}740{,}000.$$

Division can be done similarly using a table of secants.

For Brahe this method was practical because trigonometric tables accurate to fifteen places were commonly used and were sufficient for the time.

In 1590 the wedding party of James VI of Scotland (later James I of England) was en route to Denmark for the wedding of James VI and Princess Anne, the daughter of King Fredrick of Denmark. A storm forced the party ashore on the island of Mven at the observatory of Brahe. Apparently during their stay Dr. John Craig, physician to James VI, learned of the method of prosthaphaeresis and communicated it to John Napier (1550–1617), Baron of Merchiston, Scotland [1, p.342]. Napier had been pondering the problem of developing a computational aid. Upon hearing of prosthaphaeresis he redoubled his effort. Napier is given credit for having invented logarithms with the publication of *Mirifici Logarithmorum Canonis Descriptio* in 1614. However, he did not conceive of logarithms as we know them. Rather he used a correspondence between a geometric and an arithmetic progression.

Before examining Napier's development, consider how naturally the concept of a logarithm flows from a correspondence between an arithmetic and geometric progression.

Put the geometric progression 3, 9, 27, 81, 243, ... in correspondence with the arithmetic progression 2, 4, 6, 8, 10, ... as follows:

TABLE 1

$2 \longleftrightarrow 3$
$4 \longleftrightarrow 9$
$6 \longleftrightarrow 27$
$8 \longleftrightarrow 81$
$10 \longleftrightarrow 243$, and so on.

If we call each term of the arithmetic progression the logarithm of the corresponding term of the geometric progression we have the following:

TABLE 2

$$\begin{aligned} 2 &= \log 3 \\ 4 &= \log 9 \\ 6 &= \log 27 \\ 8 &= \log 81 \\ 10 &= \log 243, \text{ and so on.} \end{aligned}$$

Logarithms defined in this manner obey the usual properties. Following is a sample calculation.

Find the product, (9)(27), using logarithms as defined above and the properties of logarithms:

Solution: $\log(9)(27) = \log(9) + \log(27) = 4 + 6 = 10$. Since 10 is the logarithm of 243, we have $(9)(27) = 243$. Division and exponentiation can also be done as usual.

Note that in the table displayed above, $\log 3 = 2$, $\log 9 = 4$, etc., the concept of a "base" does not enter into the definition. However, if one is made more comfortable by annexing the notion of a base, then the table developed from the definition and the specific example given in Table 1 would clearly have the base $\sqrt{3}$. This illustrates the rather surprising fact that logarithms were not developed originally as the inverses of exponential operations.

The ability to develop logarithms from correspondences between arithmetic and geometric progressions is not unique to the examples given. If we use the geometric progression 1, 10, 100, 1000, ... and the arithmetic progression 0, 1, 2, 3, ... we could establish the integral portion of logarithms to the base 10.

The disadvantage of such systems of logarithms is that there are large "gaps" between the terms of the geometric progression. Napier solved the problem of the large gaps by simply choosing a geometric progression in which the terms were very close together. This solution was so simple that the world wondered why no one had thought of it before. The essence of Napier's development follows.

Suppose we have point b at T which is 10^7 units from point c at S at time $t = 0$ (Figure 1).

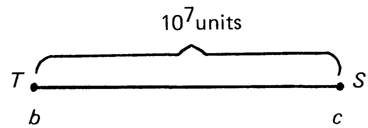

FIGURE 1

Let b move to a position which is $(1 - 1/10^7) = .9999999$ of the distance from b to c in the first unit of time, (i.e., b moves one unit). Now the distance bc is 9,999,999. In the second unit of time, move b to a position which is .9999999 of the remaining distance to c. Now the distance bc is 9,999,998.1. By continuing in this manner the distances bc form the terms of a geometric progression with a common ratio of .9999999 and a first term of 10^7. The units of time 1, 2, 3, ... form the terms of an arithmetic progression with a common difference of one and a first term of zero. By putting each distance bc in correspondence with the unit of time at which b moved to that position, we essentially have the correspondence of Napier (Figure 2).

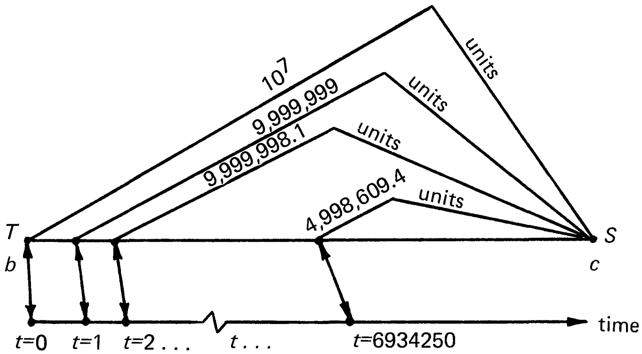

FIGURE 2

Each time, t is the logarithm of its corresponding term of the geometric progression. Equivalently, the distance from zero to t is the logarithm of the corresponding distance bc [2, p. 149].

Napier's definition of a logarithm using distances allowed the belief that $\log(x) = \log(-x)$, a belief which persisted until the time of Euler (1707–1783) [1, p. 489].

A summary of the previous results is given in Table 3.

TABLE 3

$$
\begin{aligned}
(.9999999)^0(10^7) &= 10^7\\
(.9999999)^1(10^7) &= 9{,}999{,}999\\
(.9999999)^2(10^7) &= 9{,}999{,}998.1\\
(.9999999)^3(10^7) &= 9{,}999{,}997.1\\
&\vdots\\
(.9999999)^L(10^7) &= \textit{distance bc.}
\end{aligned}
$$

The exponent L is the logarithm of Napier and the numbers 10^7, 9,999,999, 9,999,998.1, ... were called "sines" by Napier.

The laws of Napier's logarithms are different from those with which we are familiar.

If $A = (.9999999)^{L_1}(10^7)$ and $B = (.9999999)^{L_2}(10^7)$, then $AB = [(.9999999)^{L_1}(10^7)][(.9999999)^{L_2}(10^7)] = (.9999999)^{L_1+L_2}(10^7)^2$. Therefore,

$$AB/10^7 = (.9999999)^{L_1+L_2}(10^7).$$

Hence $\log(AB/10^7) = L_1 + L_2 = \log A + \log B$. It can be shown in a similar manner that $\log(A/B)(10^7) = \log A - \log B$.

The sines of Napier were very close to the terms of the geometric progression with a common ratio of .9999999 and a first term of 10^7 but not exactly the same. His "progression" was developed in the following manner.

From 10^7 subtract its 10,000,000th part and from the number obtained subtract its 10,000,000th part, and so on for one hundred numbers until 9,999,900.0004950 is reached. This gave Napier's "First Table."

The "Second Table" starts with 10^7 decreasing proportionally by the ratio of the first and last terms of the First Table for fifty numbers. The last number in the Second Table is 9,995,001.224804 which Napier mistakenly gave as 9,995,001.222927.

The "Third Table" consists of sixty-nine columns of twenty-one numbers proceeding in the proportion of the first and last numbers in the Second Table and so on until he completed his "Table of Radicals" [3, pp. 149–150].

Napier constructed the entire Table of Radicals by multiplication and subtraction, using ratios. Hence the name logarithm from the Greek logos (meaning ratio) and arithmos (meaning number). Though exponents lend themselves nicely to the idea of a logarithm, Napier did not have exponents in mind [1, p. 344].

Publication of Napier's results met with immediate acceptance. One of his most ardent admirers was Henry Briggs, first Savilian professor of geometry at Oxford. In 1615 Briggs and Napier agreed on modifying Napier's logarithms using base 10 where $\log(1) = 0$ and $\log(10) = 1$.

After Napier's death, Briggs continued to compute the logarithms of the numbers 1 through 1000, base 10, accurate to 14 decimal places. Briggs accomplished this by taking successive roots and published the results in 1617. In his book *Arithmetica Logarithmica* published in 1624 Briggs used the terminology from which we get "characteristic" and "mantissa" [1, p. 345].

Logarithms caught on very rapidly. Indeed, it has been postulated that logarithms literally lengthened the life spans of astronomers, who had formerly been sorely bent and often broken early by the masses of calculations their art required.

Somewhat later, logarithms to the base e were named natural logarithms by Mercator (1620–1687). Mercator observed that

$$\int_0^x \frac{1}{1+x}\,dx = \ln(1+x)$$

and

$$\int_0^x \frac{1}{1+x}\,dx = \int_0^x (1 - x + x^2 - x^3 + \cdots)dx$$

$$= \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

so

$$\ln(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \text{ (Mercator Series).}$$

Mercator called logarithms which could be found by the series "natural" logarithms [1, p. 423].

All that remained to be done was for Euler to give us the modern definition in terms of bases and exponents ($y = \log_b x \Leftrightarrow x = b^y, b > 0$ and $b \neq 1$) and clarify the relation between $\log(x)$ and $\log(-x)$ [1, p. 489].

REFERENCES

1. C. B. Boyer, A History of Mathematics, Wiley, New York, 1968.
2. F. Cajori, A History of Mathematics, 2nd ed., Macmillan, New York, 1919.
3. D. E. Smith, A Source Book in Mathematics, Dover, New York, 1959.